Monotone Bounded-Depth Complexity of Homomorhism Polynomials

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HOMOMORHISM POLYNOMIALS

$$\mathrm{Hom}_{H,n} = \sum_{f:V(H)\to[n]} \prod_{uv\in E(H)} x_{f(u),f(v)}$$

- Algebraic version of pattern counting problem.
- How difficult it is to compute them?

Pattern

Graph of k vertices

count number of homomorphism

$f: V(H) \to [n]$

$u,v \in E(H)$

$\Rightarrow f(u)f(v) \in E(G)$

Host

Complete graph of n vertices

Counting Homomorhism

Look for the occurrence of pattern graph in host graph.

Pattern Graph H $\longrightarrow$ Host Graph G

$\#\mathrm{Hom}(\mathcal{H})$

Exhibit a dichotomy in parameterized complexity.

Tree width of $\mathcal{H}$

constant — Easy to count

otherwise — #W[1]-hard

- Bioinfomatics
- Graph property Testing
- Captures other pattern counting problems
  eg. counting subgraphs.

Assuming Exponential-Time Hypothesis (ETH)

	Upper Bound	Lower Bound
Assuming Exponential-Time Hypothesis (ETH)	$O(n^{tw(H)+1})$	$n^{o(tw(H)/\log tw(H))}$
	[Impagliazzo, Paturi, 2001]	[Dániel Marx 2010]

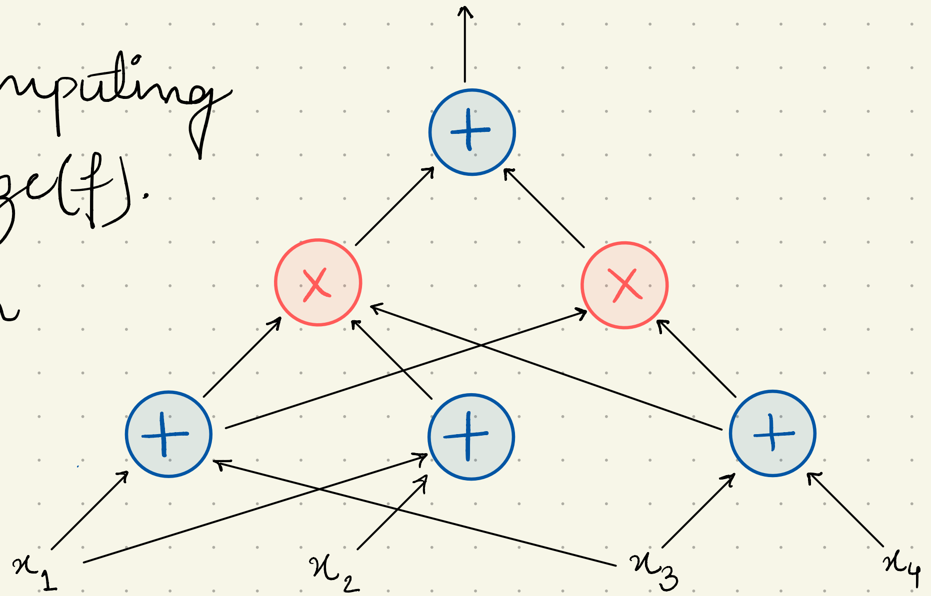
ALGEBRAIC CIRCUITS

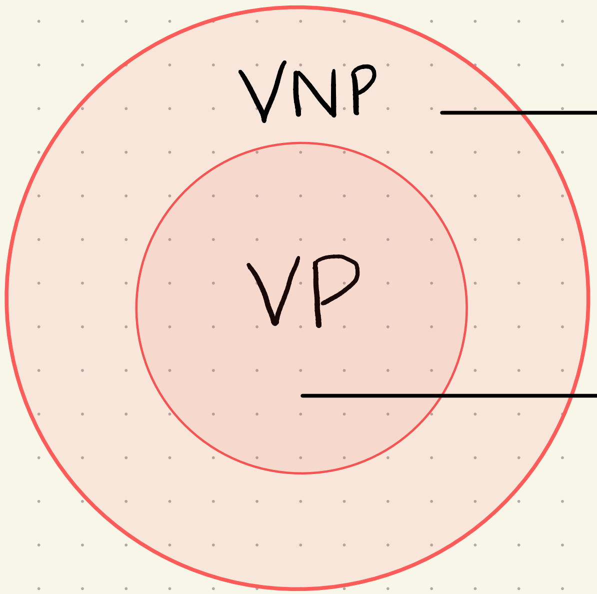
Algebraic Complexity

Size of the smallest circuit computing the polynomial. Denoted by size(f).

product-depth(f) = no of multiplication layers.

$f \in \mathbb{F}[x_1, x_2, x_3, x_4]$





Explicit polynomials coefficient can be computed efficiently.

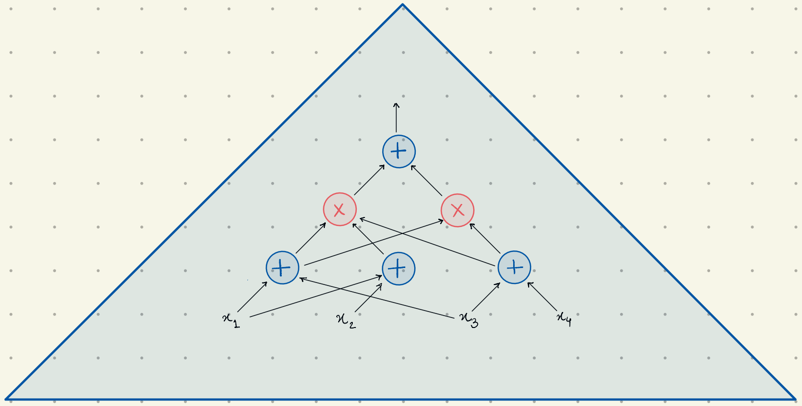
computable by circuits of size poly(n,d)

Fun Fact

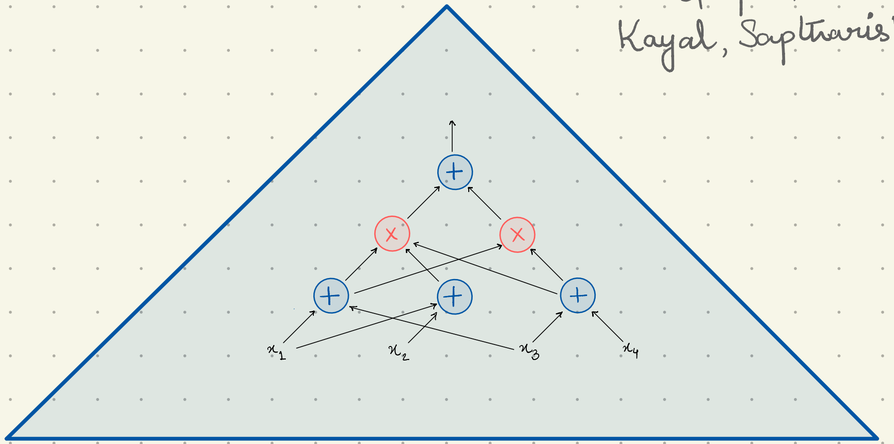
Homomorphism poly have inspired complete polynomial families for algebraic classes.

CIRCUIT LOWER BOUNDS

Gupta, Kamath
Kayal, Saptharishi '16



$\equiv$ Δ



size s circuit computing n-var
deg d polynomial

product depth Δ
size $s^{O(d^{1/2\Delta})}$

let $d = o(\log N / \log\log N)$

[Bhargav, Dutta, Saxena - 2022]

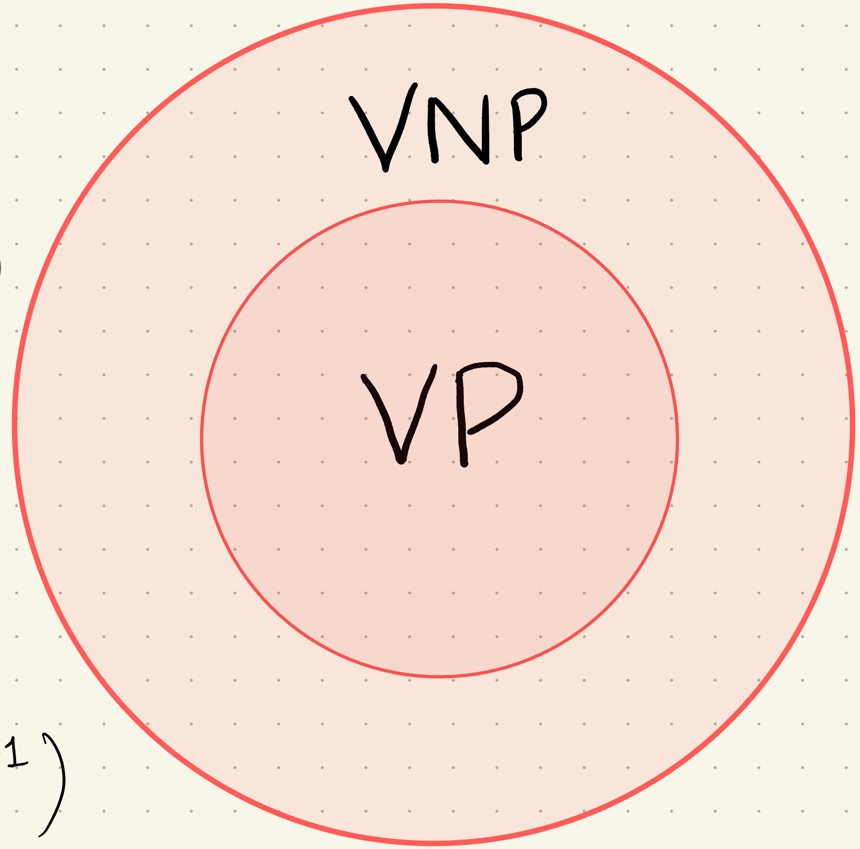
Theorem [Limaye, Srinivasan, Tavenas - 2021]
There is an explicit poly of n variate and deg d that cannot be computed by product depth Δ circuit of size

$$n^{O\left(d^{(1/F_{2\Delta})-1}/\Delta\right)}$$

$$F_n = \Theta(\phi^n) << 2^n$$

Monotone World

There are explicit polynomials which cannot be computed by small circuits. [Yehudayoff 2019, Srinivasan 2020]

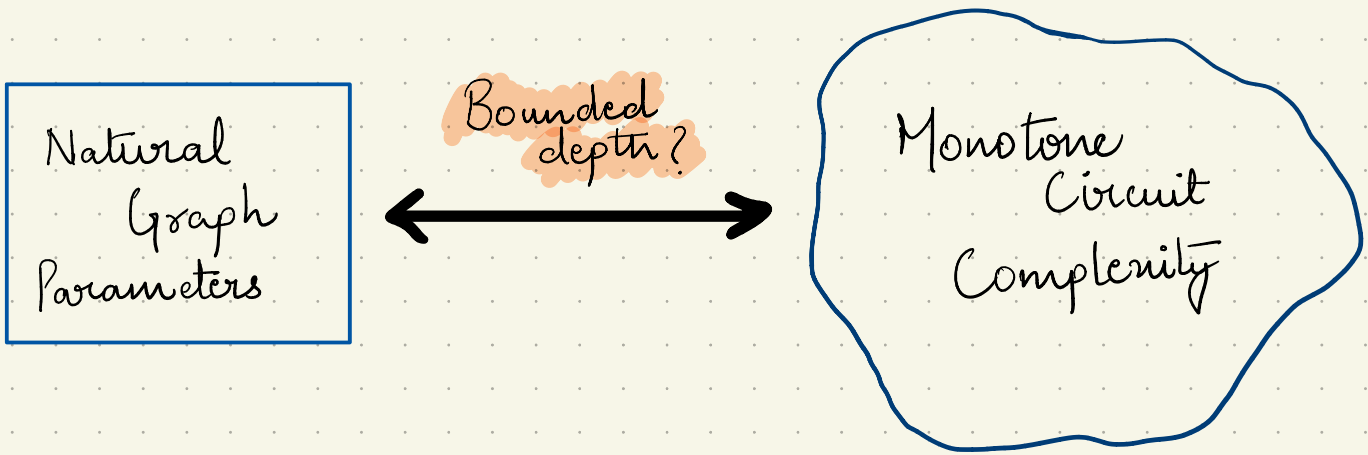


Theorem [Komarath, Pandey, Rahul - 2023]

$$\mathrm{size}(\mathrm{Hom}_{H,n}) = \Theta\left(n^{tw(H)+1}\right)$$

$$\mathrm{size}_{ABP}(\mathrm{Hom}_{H,n}) = \Theta\left(n^{pw(H)+1}\right)$$

$$\text{formula-size}(\mathrm{Hom}_{H,n}) = \Theta\left(n^{td(H)+1}\right)$$



Δ-TREEWIDTH

Tree width quantifies "Tree"-likeness.

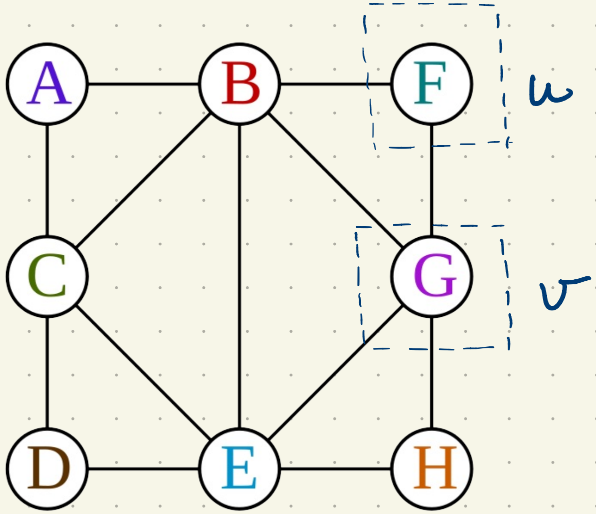
Decompose graph in a tree-like structure with properties.

$$\text{width}(T) = \max\{|X_i| : \forall i\} - 1$$

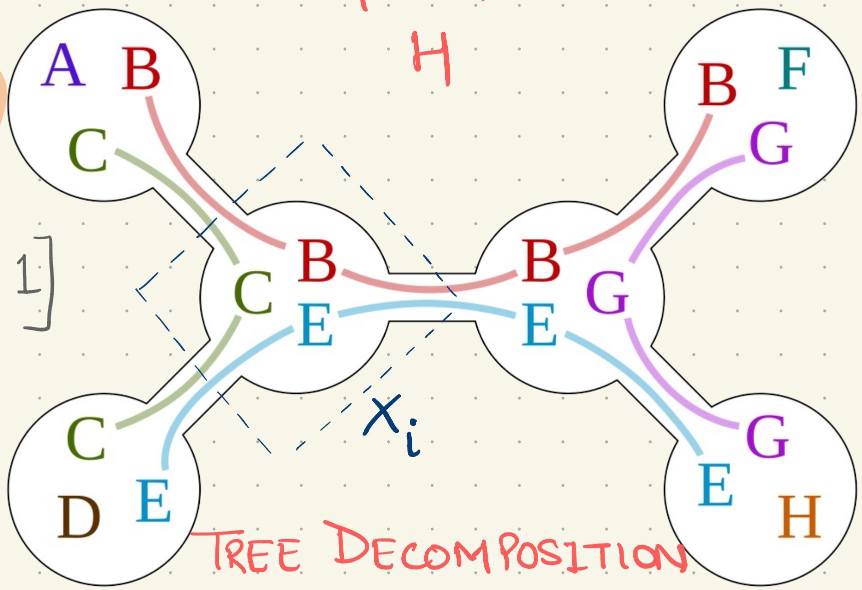
$$tw(H) = \min\left(\{\text{width}(T) : \forall T\}\right)$$

$$tw_\Delta(H) = \min\left(\left\{\begin{array}{l}\text{width}(T) : \forall T \text{ of} \\ \quad \text{height} \le \Delta\end{array}\right\}\right)$$

[Single node is height 1]



GRAPH H



TREE DECOMPOSITION T

Properties:

1) $\bigcup_i X_i = V$

2) $\{X_i : u \in X_i\}$ forms a connected subtree.

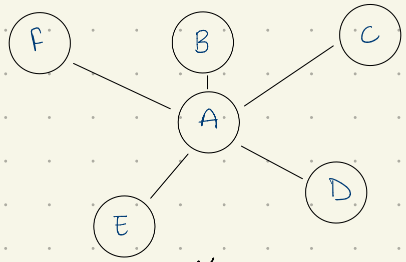
3) $\forall (u,v) \in E(G), \exists i, \{u,v\} \subseteq X_i$

[Picture Credits: Wikipedia]

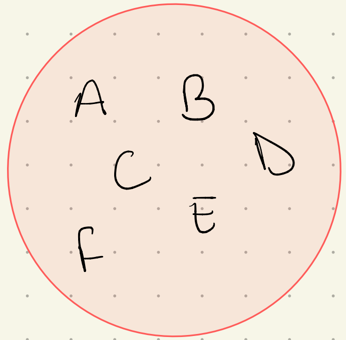
Δ-TREEWIDTH

$$tw_{\Delta}(H) = \min\left(\left\{ \begin{array}{l} width(T) : \forall\, T \text{ of} \\ \text{height} \leq \Delta \end{array} \right\}\right)$$

[Single node is height 1]



$K_{1,5}$



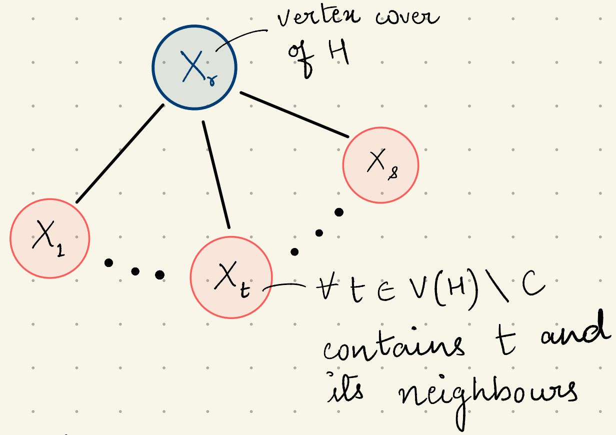
$tw_1(K_{1,5}) = 6$

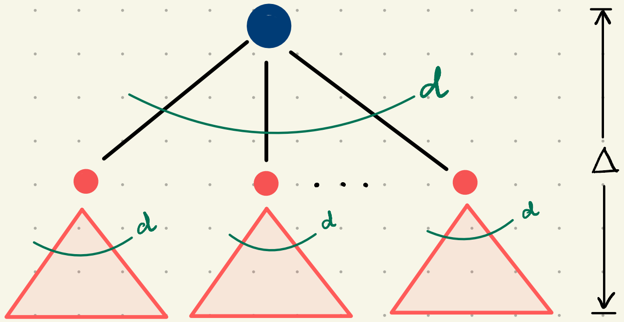
Let $k = |V(H)|$

$$tw_1(H) = k \quad , \quad \text{and} \quad tw_k(H) = tw(H)$$

Lemma: Let C be the vertex cover of H

$tw_2(H) \leq vc(H)$, vertex-cover number





Theorem: [This work]

Let T_Δ be the full d-ary tree of height Δ

- $tw_\Delta(T_\Delta) = 1$
- $tw_{\Delta-1}(T_\Delta) \geq d-1$

BOUNDED TW $\longrightarrow$ CIRCUITS

Fix a pattern graph H

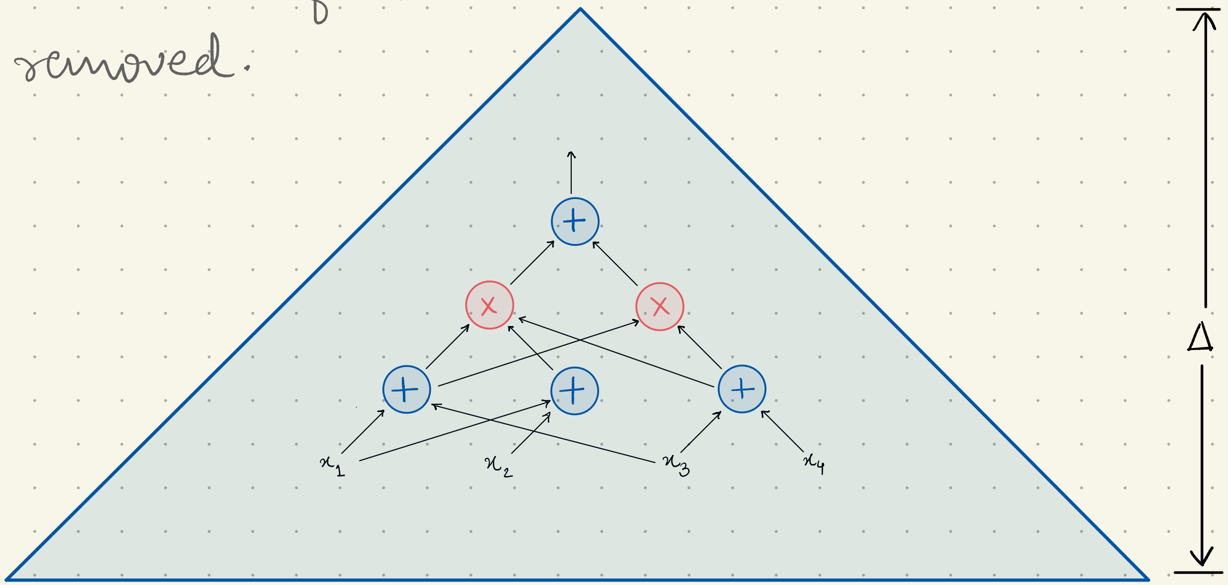
Δ, n are natural numbers

$ptw_\Delta(H) = tw_\Delta(H')$ deg 1 vertices of H are removed.

Theorem [This work]

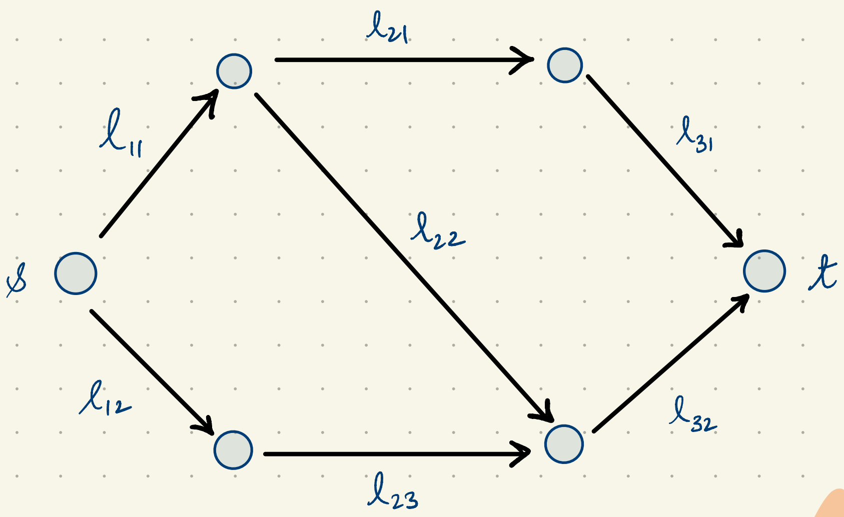
Monotone depth Δ circuit complexity of $\mathrm{Hom}_{H,n}$ is

$$\Theta\left(n^{ptw_\Delta(H)+1}\right)$$



Trivial poly(n) size depth-2 circuit. But ptw_Δ gives precise exponent.

ALGEBRAIC BRANCHING PROGRAMS



$$f = \sum_{\text{path } \gamma : s \rightsquigarrow t} wt(\gamma)$$

product of linear poly on edge weights

$$f(x_1, \ldots, x_n) = \mathrm{Det}\begin{pmatrix} & & \vdots & & \\ \cdots & & a x_i + b & & \cdots \\ & & \vdots & & \end{pmatrix}_{w \times w}$$

Tree decomposition is a path.

Path width governs the monotone ABP complexity of $\mathrm{Hom}_{H,n}$

Length Bounded versions are $pw_\Delta(H)$.

Theorem [This work]

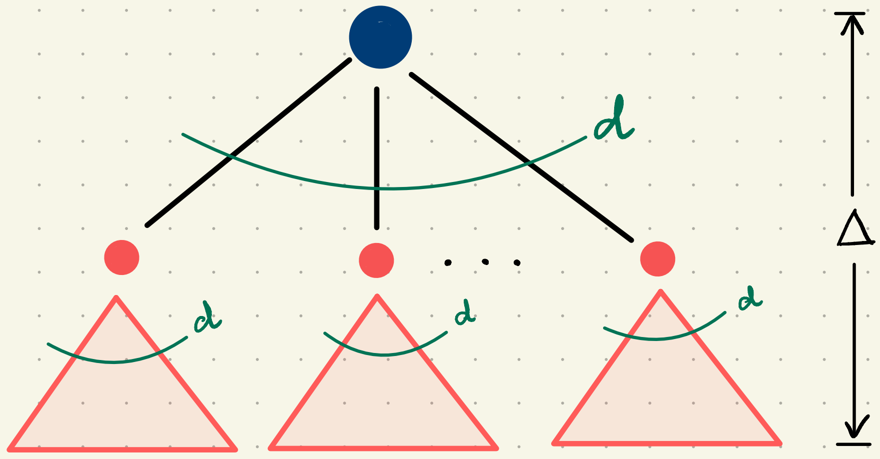
Monotone length Δ ABP complexity of $\mathrm{Hom}_{H,n}$ is

$$\Theta\left(n^{ppw_\Delta(H)+1}\right)$$

pruned version.

MONOTONE DEPTH HIERARCHY

$$\mathrm{ColSub}_{H,n} = \sum_{f: V(H) \to [n]} \prod_{uv \in E(H)} x^{(uv)}_{f(u),f(v)}$$



T_Δ

Theorem [This work]

For any natural numbers n, Δ

Pattern graph H_Δ of size $\Theta(n)$

$$\widetilde{\mathrm{size}}_{\Delta+1}(\mathrm{ColSub}_{H,n}) = \mathrm{poly}(n)$$

$$\widetilde{\mathrm{size}}_{\Delta}(\mathrm{ColSub}_{H,n}) = n^{\Omega(n^{1/\Delta})}$$

$$H_\Delta = T_{\Delta+2}$$

- Classical depth reduction results prove this is optimal.
- Near optimal hierarchy results were known for small $\Delta = o(\log n / \log\log n)$ Chillara, Engels, Limaye, Srinivasan 2018

CONCLUSION

- Characterized bounded depth (length) monotone circuit (ABP) complexity of Hom

  $\Theta(n^{ptw(H)+1})$, $\Theta(n^{ppw(H)+1})$

- Monotone depth Hierarchy

Open Questions

- Prove LST type bounds $n^{O(d^{(1/f_{2\Delta})-1}/\Delta)}$
- Hierarchy theorem using pathwidths for ABP